

# Can binary mergers produce maximally spinning black holes?

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(Dated: July 2008)

Gravitational waves carry away both energy and angular momentum as binary black holes inspiral and merge. The relative efficiency with which they are radiated determines whether the final black hole of mass  $M_f$  and spin  $S_f$  saturates the Kerr limit ( $\chi_f \equiv S_f/M_f^2 \leq 1$ ). Extrapolating from the test-particle limit, we propose expressions for  $S_f$  and  $M_f$  for mergers with initial spins aligned or anti-aligned with the orbital angular momentum. We predict the the final spin at plunge for equal-mass non-spinning binaries to better than 1%, and that equal-mass maximally spinning aligned mergers lead to nearly maximally spinning final black holes ( $\chi_f \simeq 0.9988$ ). We also find black holes can always be spun up by aligned mergers provided the mass ratio is small enough.

Recent breakthroughs in numerical relativity have led to successful simulations of the inspiral, merger, and ringdown of binary black holes (BBHs) [1, 2, 3]. Despite this progress, approximations underlying these simulations limit them to mergers with large but submaximal initial spins ( $\chi_i \lesssim 0.9$ ) and order-unity mass ratios ( $q \equiv m_2/m_1 \geq 1/6$ ). Yet the maximally spinning regime is of considerable theoretical and observational interest. Maximally spinning black holes barely manage to hide their singularities within their event horizons, and observable, “naked” singularities are expected for spins  $\chi > 1$ . While infinitesimal processes like steady accretion cannot produce naked singularities [4], Penrose’s cosmic-censorship conjecture [5] that such singularities can *never* be created has not been proven for comparable-mass BBH mergers. Black-hole spins are not a purely theoretical concern; they can be measured by reverberation mapping of iron  $K\alpha$  fluorescence in the spectra of active galactic nuclei (AGN) [6]. This technique has been applied to *XMM-Newton* observations of the Seyfert 1.2 galaxy MCG-06-30-15, leading to a measured spin  $\chi = 0.989^{+0.009}_{-0.002}$  very near the maximal limit [7]. The spins of supermassive black holes (SBHs) also offer important insights into their formation. Some expect that highly spinning SBHs will only be found in gas-rich systems like spirals [8]. Others suggest that gas accretion occurs through a series of chaotically oriented episodes, leading to moderate spins ( $\chi \sim 0.1\text{--}0.3$ ) lower than those expected from comparable-mass mergers in gas-poor ellipticals [9]. Comparing measured spins in spirals and ellipticals will distinguish between these two scenarios.

In the absence of reliable simulations of maximally spinning BBH mergers, we must rely on various approximations in this important regime. Hughes and Blandford (hereafter HB) [10] assumed that the energy and angular momentum radiated during the inspiral stage dominates that radiated during the comparatively brief merger and ringdown. They then used conservation of energy and angular momentum to equate these quantities when the BBHs reach their innermost stable circular orbit (ISCO)

to the mass and spin of the final black hole,

$$M_{f,\text{HB}} = m_1 + m_2 E(\chi_1), \quad (1a)$$

$$S_{f,\text{HB}} = m_1 m_2 L_{\text{orb}}(\chi_1) + m_1^2 \chi_1. \quad (1b)$$

Here  $E(\chi)$  is the energy per unit mass of a test particle on an equatorial orbit of a Kerr black hole with spin parameter  $\chi$  and  $L_{\text{orb}}(\chi)$  is the corresponding dimensionless orbital angular momentum. While  $E(\chi)$  and  $L_{\text{orb}}(\chi)$  are only available in the test-particle limit ( $m_2 \ll m_1$ ) [11], HB extended these exact, analytic results to comparable-mass mergers. While this approach reproduces the test-particle limit by design, it is not symmetric under the exchange of labels “1” and “2” and leads to supermaximal spins ( $\chi_f > 1$ ) for near maximal mergers.

Buonanno, Lehner, and Kidder [12] (hereafter BKL) remedied these problems by (1) conserving mass, (2) including the initial spin of the smaller black hole, and (3) using the spin parameter of the *final* black hole  $\chi_f$  rather than that of the more massive of the initial BBHs  $\chi_1$  to determine the orbital angular momentum,

$$M_{f,\text{BKL}} = m_1 + m_2, \quad (2a)$$

$$S_{f,\text{BKL}} = m_1 m_2 L_{\text{orb}}(\chi_f) + m_1^2 \chi_1 + m_2^2 \chi_2. \quad (2b)$$

$L_{\text{orb}}(\chi_f)$  in Eq. (2b) corresponds to a prograde (retrograde) equatorial orbit if the final spin is aligned (anti-aligned). This approach agrees remarkably well with existing simulations of moderately spinning, comparable-mass BBH mergers, but the assumption of mass conservation reduces agreement with the test-particle limit to zeroth order in the mass ratio  $q$ . While this approximation is strictly valid to 10% as noted by BKL, it artificially reduces  $\chi_f$  in the important highly spinning regime.

To improve on BKL, we seek an analytic expression for the final mass  $M_f$  for arbitrary mass ratios and initial spins. In the spirit of their work, we propose

$$M_{f,\text{K}} = M - \mu[1 - E(\chi_f)], \quad (3)$$

where  $M \equiv m_1 + m_2$  is the total initial mass and  $\mu \equiv m_1 m_2 / M$  is the reduced mass. This proposal both agrees

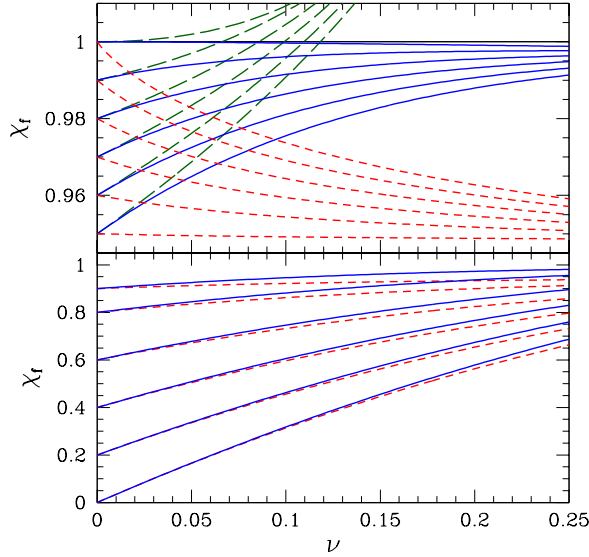


FIG. 1: The final spin parameter  $\chi_f$  for mergers with equal initial spin parameters ( $\chi_1 = \chi_2$ ) and mass ratio  $\nu$ . Each curve gives a different value of  $\chi_i$ , which can be read off the y-axis as  $\chi_f = \chi_i$  for  $\nu = 0$ . Curves in blue (solid) denote our predictions, red (short-dashed) those of BKL, and green (long-dashed) those of HB. *Top panel*: the highly spinning regime ( $\chi_i \geq 0.95$ ). *Bottom panel*: aligned spins ( $\chi_i \geq 0$ ).

with Eq. (1a) to first order in  $q$  and is symmetric under exchange of the black hole labels “1” and “2”. Dividing Eq. (2b) by the square of (3), we find

$$\chi_f = \frac{\nu L_{\text{orb}}(\chi_f) + \frac{\chi_1}{4}(1 + \sqrt{1 - 4\nu})^2 + \frac{\chi_2}{4}(1 - \sqrt{1 - 4\nu})^2}{\{1 - \nu[1 - E(\chi_f)]\}^2}, \quad (4)$$

for the final spin parameter, where  $\nu \equiv \mu/M$ .

In Fig. 1, we compare this new prediction for  $\chi_f$  (hereafter K) with those of HB (after including  $S_2$  in Eq. (1b)) and BKL. We consider BBHs with equal initial spins ( $\chi_1 = \chi_2 = \chi_i$ ) aligned with the orbital angular momentum. All three approaches reproduce the trivial zeroth order result that  $\chi_f = \chi_i$  in the test-particle limit ( $\nu = 0$ ). However, only K and HB provide the correct first-order behavior (the value of  $\partial\chi_f/\partial\nu$ ) in this limit [22]. The BKL assumption  $M_f = M$  implies that  $E = 1$  and makes the denominator of Eq. (4) equal unity. Overestimating  $M_f$  to first order in  $\nu$  leads to a first-order underestimate in  $\chi_f = S_f/M_f^2$ , accounting for the negative values of  $\partial\chi_f/\partial\nu$  given by BKL for  $\chi_i \gtrsim 0.95$ . K and HB predict  $\partial\chi_f/\partial\nu(\chi_i \leq 1) \geq 0$ , with equality for maximal initial spins. This agrees with the famous result of Bardeen [13] that black holes can be spun up to the Kerr limit  $\chi = 1$  after accreting a finite mass of test particles. Despite its success for  $\nu = 0$ , HB predicts naked singularities ( $\chi_f > 1$ ) for comparable-mass mergers.

Let’s examine the predictions of K more closely in the

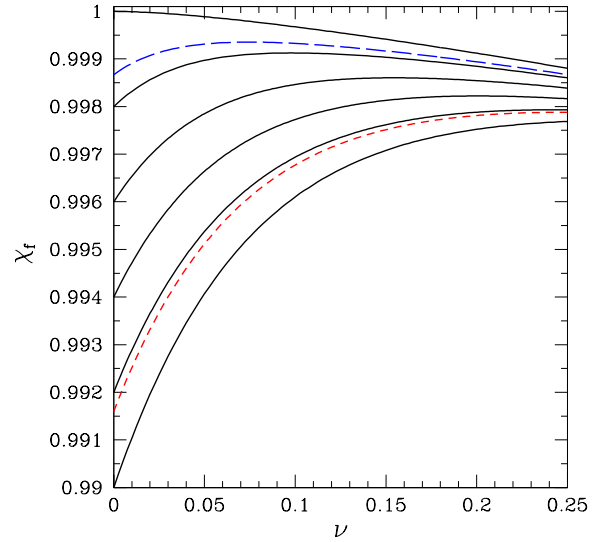


FIG. 2: The final spin parameter  $\chi_f$  as a function of  $\nu$  for the merger of BBHs with equal and nearly maximal initial spin parameters ( $\chi_1 = \chi_2 > 0.99$ ). The red (short-dashed) curve shows the maximum initial spin ( $\chi_i \simeq 0.9916$ ) for which this function monotonically increases with  $\nu$ . The blue (long-dashed) curve shows the maximum initial spin ( $\chi_i \simeq 0.9987$ ) for which black holes are spun up by equal-mass mergers.

limit of nearly maximal spins ( $\chi_1 = \chi_2 > 0.99$ ). In Fig. 2, we show  $\chi_f(\nu)$  at intervals of 0.002 in  $\chi_i$ . For  $\chi_i \lesssim 0.9916$  (the red, short-dashed curve),  $\chi_f(\nu)$  is monotonically increasing implying that equal-mass mergers ( $\nu = 0.25$ ) are the most efficient way of spinning up a black hole. As  $\chi_i$  increases, the maximum  $\nu_{\text{max}}$  of  $\chi_f(\nu)$  moves to lower  $\nu$ . For  $\chi_i \gtrsim 0.9987$  (the blue, long-dashed curve) equal-mass mergers spin *down* black holes rather than spinning them up. Finally, at  $\chi_i = 1$ , the maximum reaches  $\nu = 0$  making  $\chi_f(\nu)$  monotonically decreasing. Thus maximally spinning black holes will be spun down for all mergers with  $\nu > 0$ . The merger of maximally spinning equal-mass BBHs yields a black hole with  $\chi_f \simeq 0.9988$ .

This behavior is summarized in Fig. 3. At  $\chi_i \gtrsim 0.9916$ ,  $\nu_{\text{max}}$  falls below 0.25 and the red curve peels away from the blue curve in the top panel, indicating that equal-mass mergers are no longer the most efficient way of spinning up highly spinning black holes. While the blue curve belonging to equal-mass mergers falls below the  $\chi_f = \chi_i$  reference curve for  $\chi_i \gtrsim 0.9987$ , the red curve always has  $\chi_f \geq \chi_i$  implying that BBH mergers can *always* spin up black holes provided  $\nu$  is low enough. Though the specific values provided here are only as accurate as the approximation itself, we believe this qualitative behavior will ultimately be verified by numerical simulations once near maximally spinning initial data can be constructed.

Let’s compare these predictions to recent numerical-

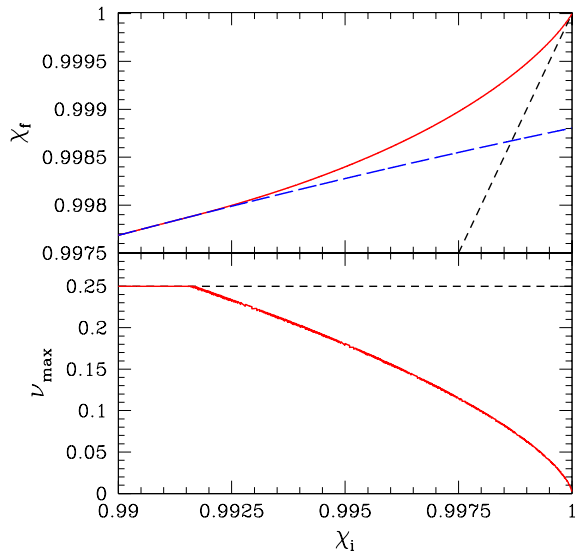


FIG. 3: *Top panel:* The final spin parameter  $\chi_f$  as a function of  $\chi_i$  for the merger of BBHs with equal and nearly maximal initial spin parameters ( $\chi_1 = \chi_2 = \chi_i$ ). The blue (long-dashed) curve corresponds to equal-mass BBHs, the red (solid) curve to mergers with mass ratio  $\nu_{\max}$  that maximizes  $\chi_f(\nu)$ , and the black (short-dashed) curve to the reference  $\chi_f = \chi_i$ . *Bottom panel:* The value  $\nu_{\max}(\chi_i)$  described above.

relativity simulations. We begin with initially non-spinning BBHs of mass ratio  $\nu$ . The top panel of Fig. 4 shows the final masses  $M_f$  determined from the energy radiated (squares), apparent horizon (triangles), and quasi-normal modes (QNMs, circles) [14]. Eq. (3) underestimates  $M_f$  by nearly a factor of two for equal-mass mergers, possibly indicating that as much as 50% of the energy is radiated after ISCO for these mergers. Better agreement is seen at lower mass ratios, in line with arguments that the energy radiated during merger and ringdown scales as  $\nu^2$  rather than  $\nu$  [14, 15]. Despite these poor estimates of  $M_f$ , our predicted final spins  $\chi_f$  are amazingly accurate; the equal-mass prediction ( $\chi_f = 0.687$ ) lies within the numerical errors ( $\Delta\chi_f \simeq 0.002$ ) of most simulations, an order-of-magnitude improvement over BKL. This is telling us something profound; while significant energy and angular momentum are radiated after plunge, they are radiated in the appropriate ratio to preserve the spin  $\chi_f = S_f/M_f^2$ .

This is not always the case for spinning BBHs. Fig. 5 shows the final masses and spins for equal-mass, equal-spin BBH mergers. If we believe our approximation, as much as 45% of the energy is radiated after the inspiral stage for aligned, highly spinning configurations. This energy loss tends to drive  $\chi_f$  up, but apparently angular momentum is radiated even more efficiently after plunge because the numerically determined spins are below our

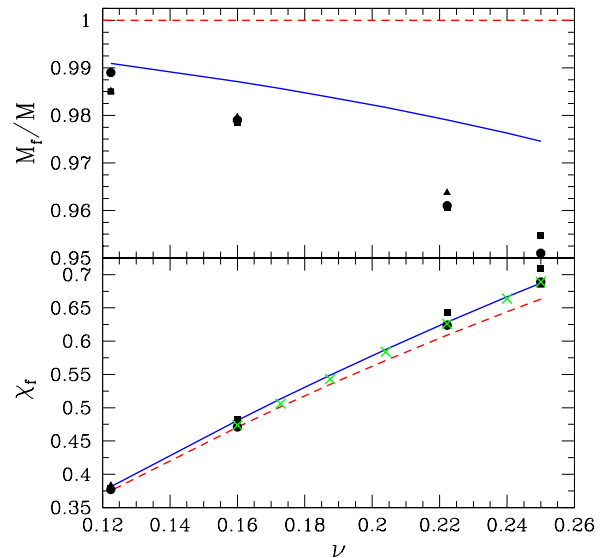


FIG. 4: *Top panel:* The final mass  $M_f(\nu)$  for initially non-spinning mergers. The red (short-dashed) curve is the BKL approximation, while the blue (solid) curve is that of Eq. (3). The square, triangle, and circular points are three numerical estimates of  $M_f$  [14]. *Bottom panel:* The final spin  $\chi_f(\nu)$  for non-spinning mergers. The red (short-dashed) curve is the BKL prediction; the blue (solid) curve is that of Eq. (4). The black points are three estimates of the final spins from [14], while the green Xs are simulated final spins from [15].

predictions. The BKL approximation actually does better for these configurations, but this appears to be a conspiracy between the excess final mass compensating for the failure to account for the angular momentum radiated after plunge. Angular momentum is radiated less efficiently after plunge for anti-aligned spins, leading us to underestimate  $\chi_f$  for  $\chi_i < 0$ . Future work will attempt to improve our approximation by accounting for radiation after the ISCO.

This same trend is seen in equal-mass, unequal-spin simulations [18]. There is excellent agreement between our approach and all simulations with vanishing *total* initial spin like those shown in the upper left panel of Fig. 6. Our approach improves on BKL except for the highly spinning aligned configurations where their artificially high  $M_f$  drives  $\chi_f$  down.

BKL used conservation of angular momentum to predict with great success the final spin in BBH mergers. We have improved on their approach by allowing for energy loss during the inspiral, and have uncovered qualitatively new dependence of the final spin on the spins and mass ratio of the initial BBHs. Equal-mass BBH mergers might lead to higher final spins than expected, rivaling the highest spins ( $\chi \simeq 0.998$ ) allowed by physical accretion flows [20]. Black holes can *always* be spun up by

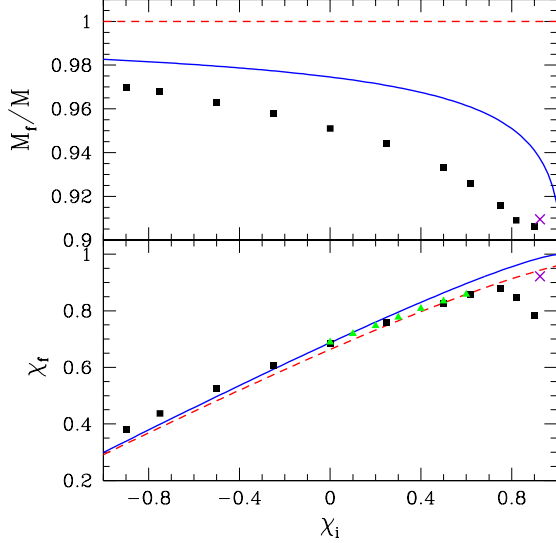


FIG. 5: *Top panel:* The final mass  $M_f(\chi_i)$  for equal-mass, equal-spin mergers. The red (short-dashed) curve is the BKL prediction, while the blue (solid) curve is that of Eq. (3). Square points are simulations from [16]; the purple X is from [17]. *Bottom panel:* The final spin parameter  $\chi_f(\nu)$  for these mergers. The red (short-dashed) curve is the BKL prediction, the blue (solid) curve is that of Eq. (4). Black squares, purple Xs, and green triangles are from [16], [17], and [18].

aligned mergers provided the mass ratio is small enough. We have found that for non-spinning BBH mergers, exact analytic results for test particles on Kerr geodesics can be extrapolated all the way to equal masses with better than 1% accuracy. Our approach provides quantitative predictions for regimes not yet accessible to numerical relativity, and helps guide the choice of simulations to explore the 7-dimensional parameter space of BBH mergers formed by  $\nu$ ,  $\vec{S}_1$ , and  $\vec{S}_2$ . This complements fitting formulae that successfully describe final spins for comparable-mass mergers [18, 19, 21]. It can help identify where in parameter space black holes will be spun up by mergers, and where the final spin will vanish leading to interesting orbital dynamics. Understanding BBH mergers is an essential goal of astrophysics, general relativity, and gravitational-wave detection, and can only be achieved by combining numerical relativity, post-Newtonian techniques, and analytic approximations like ours.

**Acknowledgements.** We thank Emanuele Berti, Latham Boyle, Avery Broderick, Alessandra Buonanno, Larry Kidder, Luis Lehner, Harald Pfeiffer, Denis Pollney, and Luciano Rezzolla for useful conversations.

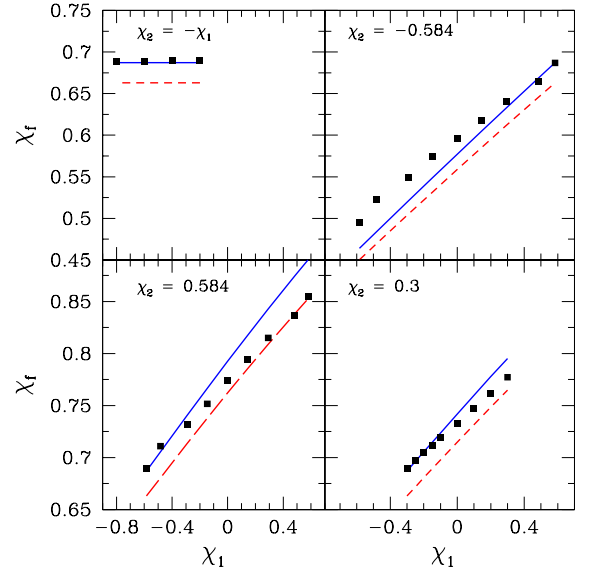


FIG. 6: Final spin  $\chi_f$  as a function of  $\chi_1$  for the four sequences of equal-mass, unequal-spin simulations presented in [18]. The value of  $\chi_2$  for each sequence is listed on each panel, and we have exchanged  $\chi_1$  and  $\chi_2$  in the upper right panel for convenient presentation. Blue (solid) curves are our predictions, red (dashed) curves those of BKL.

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- [22] Correct dependence on  $\nu$  helps calibrate fits for  $\chi_f$ ; for example, Eq. (4) implies the coefficients of [19] should be  $s_4 = -(2\sqrt{3})/9$ ,  $t_0 = -(16\sqrt{3})/9$ , and  $t_1 = 2\sqrt{3}$ .